NonEq. Stat Physics

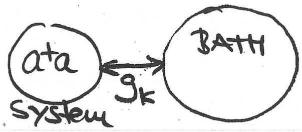
Lecture 10

[Reference: Gardiner Collet PRA 1984 Input output relations in damped quantum systems]

Quantum Langevin Equations Input & Output formalism

Last lecture the quantum Langevin Equation was derived in the form.

$$\boxed{\partial_t \hat{a} = -\underbrace{\kappa/2\, \hat{a}}_{\text{Add damping}} + i\omega_0 \hat{a} + \underbrace{\hat{f}(t)}_{\text{Langevin operator}}}$$



where the system Hamiltonian was

$$\hat{H}_{sys} = \hbar\omega_0 a^\dagger a + \frac{\hbar\omega_0}{2} \qquad \hat{H}_{int} = \sum_k g_k \hat{b}_k^\dagger \hat{a} + h.c \qquad H_{BATH} = \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k$$

we found that

$$\boxed{\begin{aligned} \langle \hat{f}(t), \hat{f}^\dagger(t')\rangle &= \kappa(\bar{n}(\omega)+1)\,\delta(t-t') \\ \langle \hat{f}^\dagger(t), f(t')\rangle &= \kappa(\bar{n}(\omega))\,\delta(t-t') \end{aligned}} \quad (*)$$

with $\kappa = 2\pi D(\omega_0)|g(\omega_0)|^2$
$\underbrace{}_{\text{constant coupling (vs. frequency)}}$

and $\bar{n}(\omega) = \langle b_k^\dagger(0)\, b_k(0)\rangle$ Note that (*) yields $\kappa = \frac{\hbar\omega}{\bar{n}(\omega)} \int \langle F(t) F(t')\rangle\, dt'$

FLUCTUATION DISSIPATION THEOREM.

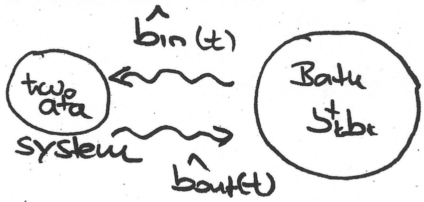
The assumptions were:

- infinitely closely spaced levels
- coupling g_k approximately constant near ω_0 (Markov Approximation)
- system does not change the state of the bath.

Input output relations in damped quantum systems

- Theory distinguishes input from output noise (i.e. the "direction" of $\hat{f}$)
- Theory is a formalism in which the langevin force $\hat{f}$ are INPUT NOISES of INPUT MODES.
- Theory has a time reversed Quantum Langevin equation in which Langevin force is on OUTPUT MODE.

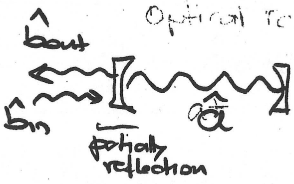
conceptually



goal: relate input noises to output noises to each other. we will show that there exist a relation between them.

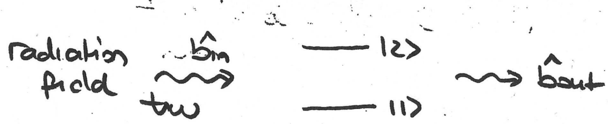
Example: Quantum Optics:

OPTICAL RESONATOR:



(e.g Haroche Nobel Prize 2019)

TWO LEVEL SYSTEM:



(Physics Nobel 2014 Wineland)

Input-output theory (gardiner & collet) of open quantum systems

$$\boxed{\begin{aligned}&\hat{H} = \hat{H}_{SYS} + \hat{H}_{BATH} + \hat{H}_{int} && H_{sys} = \hbar\omega_0 \hat{a}^\dagger \hat{a}\\ &\hat{H}_B = \hbar \int_{-\infty}^{\infty} d\omega\, \omega\, \hat{b}^\dagger(\omega)\hat{b}(\omega) && \text{Bath Modes}\\ &\hat{H}_{int} = i\hbar \int_{-\infty}^{\infty} d\omega\, \kappa(\omega)\,[\hat{b}^\dagger(\omega)\cdot\hat{c} - \hat{c}^\dagger \hat{b}(\omega)]\end{aligned}}$$

$\hat{c}$: system Operator describing coupling to bath (e.g. $\hat{c}=\hat{a}$) for energy relaxation (jump operator)

Commutator relations of the continously distributed bath modes

$$[\hat{b}(\omega), \hat{b}^\dagger(\omega')] = \delta(\omega-\omega')$$

Derivation of the Q.L.E. Heisenberg Equation of motion

$$d_t \hat{b}(\omega) = -\frac{i}{\hbar}[\hat{b}(\omega), \hat{H}] = -i\omega\, \hat{b}(\omega) + \kappa(\omega)\hat{c}(t)$$

$$d_t \hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}] = -i\omega \hat{a}(t) + \int_{-\infty}^{\infty} d\omega\, \kappa(\omega)\left\{\hat{b}^\dagger(\omega)[\hat{a},\hat{c}] - [\hat{a},\hat{c}^\dagger]\hat{b}(\omega)\right\}$$

As before we integrate and solve bath mode (from $t=t_0$)

$$\boxed{\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + \kappa(\omega)\cdot\int_{t_0}^{t} e^{-i\omega(t-t')}\hat{c}(t')\,dt' \qquad (\text{Eq I})}$$

(Variation of constant)

(Forward solution)

Next insert into equation for $\hat{a}$:

$$d_t\hat{a} = -i\omega_0\hat{a} + \int d\omega\, \kappa(\omega)\left\{e^{-i\omega(t-t_0)}\hat{b}_0^\dagger(\omega)[\hat{a},\hat{c}] - [\hat{a},\hat{c}^\dagger]e^{-i\omega(t-t_0)}\hat{b}_0(\omega)\right\}$$

$$+ \int d\omega\, |\kappa(\omega)|^2 \int_{t_0}^{t}\left(e^{i\omega(t-t')}\hat{c}^\dagger(t')[\hat{a},\hat{c}] - [\hat{a},\hat{c}^\dagger]e^{-i\omega(t-t')}\hat{c}(t')\right)dt'$$

approximations 1st Markov $\boxed{\kappa(\omega) \equiv \sqrt{\gamma/2\pi}}$ frequency independent

use:

$$\boxed{\begin{aligned}&\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')} = 2\pi\,\delta(t-t')\\ \text{and}\quad &\int_{t_0}^{t} \delta(t-t')\,\hat{c}(t')\,dt' = \frac{\hat{c}(t)}{2}\end{aligned}}$$

$[a,a^\dagger]=1$

This yields for $\hat{c}=\hat{a}$; since

$$\boxed{d_t\hat{a} = -i\omega_0\hat{a} - \frac{\gamma}{2}\hat{a} - \sqrt{\gamma}\cdot\underbrace{\int d\omega\, e^{-i\omega(t-t_0)}\hat{b}_0(\omega)}_{\text{Define this as INPUT MODE}}}$$

since: 2nd Term $\int_{t_0}^{t}\int d\omega\,|\kappa(\omega)|^2 e^{-i\omega(t-t')}\hat{c}(t')\,dt' \cong |\kappa(\omega)|^2\int_{t_0}^{t}\delta(t-t')\,\hat{c}(t')\,dt'$

$$= \underbrace{|\kappa(\omega)|^2\, 2\pi}_{\hat{=}\ \gamma\ :\ \text{decay}}\ \frac{\hat{c}(t)}{2}$$

Input Output Theory of Open Quantum Systems.

$$\hat{b}_{in}(t) \equiv \int d\omega\, e^{-i(\omega(t-t_0))} \hat{b}_o(\omega) \qquad \text{Input mode}$$

thus

$$\boxed{d_t \hat{a} = -i\omega_0 \hat{a} - \frac{\gamma}{2}\hat{a} - \sqrt{\gamma}\, \hat{b}_{in}(t)} \quad \text{Forward Q.L.E.}$$

alternative form: (use Eq I)

$$\int \hat{b}(\omega)\, d\omega = \underbrace{\int \hat{b}_o(\omega)\, e^{i\omega(t-t_0)}}_{\hat{b}_{in}(t)} + \underbrace{\kappa(\omega) \iint e^{i(\omega(t-t'))}\, \hat{c}(t')\, dt\, d\omega}_{\hat{=}\ \frac{\hat{c}(t)}{2} \cdot \sqrt{\gamma}}$$

Thus:

$$\boxed{\int \hat{b}(\omega)\, d\omega = \hat{b}_{in}(t) + \frac{\hat{c}(t)}{2}\sqrt{\gamma}} \quad (\text{Eq II})$$

Next let us assume time reversed Langevin equations (noise o/input thus turn to outputs)

we write for the bath mode $\hat{b}(\omega)$ by integrating from $(t \ldots t_1)$ $t_1 > t$

$$\boxed{\hat{b}(\omega) = e^{-i\omega(t-t_1)}\, \hat{b}_1(\omega) - \kappa(\omega) \int_t^{t_1} e^{-i(\omega)(t-t')}\, \hat{c}(t')\, dt'} \quad \hat{b}_1(\omega) = \hat{b}(\omega)_{t=t_1}$$

(note sign change)

compare to $b_o(\omega) = b(\omega)_{t=t_0}$

Solving equation for the case of $\hat{c} = \hat{a}$ (energy relaxation) yields:

$$\boxed{d_t \hat{a}(t) = -i\omega_0 \hat{a} + \gamma/2\, \hat{a} + \sqrt{\gamma} \underbrace{\int d\omega\, e^{-i\omega(t-t_1)}\, \hat{b}_1(\omega)}_{\text{OUTPUT FIELD / NOISE}}}$$

backward QUANTUM LANGEVIN EQS.

$$\hat{b}_{out}(t) = \int d\omega\, e^{-i\omega(t-t_1)}\, \hat{b}_1(\omega)$$

Note that this TIME REVERSED LANGEVIN EQS OBEYS

$$\boxed{\begin{aligned} \hat{b}_{in}(t) &\rightarrow \hat{b}_{out}(t) \\ \sqrt{\gamma} &\rightarrow \sqrt{\gamma} \\ \frac{\gamma}{2} a &\rightarrow -\frac{\gamma}{2} a \end{aligned}}$$

$$\boxed{\begin{aligned} &\text{Forward Q.L.E} \quad d_t \hat{a} = -i\omega_0 \hat{a} - \frac{\gamma}{2}\hat{a} - \sqrt{\gamma}\, \hat{b}_{in} \\ &\text{Backward Q.L.E.} \quad d_t \hat{a} = -i\omega_0 \hat{a} + \frac{\gamma}{2}\hat{a} + \sqrt{\gamma}\, \hat{b}_{out} \end{aligned}}$$

hence $\int d\omega\, b(\omega) = b_{out}(t) - \sqrt{\frac{\gamma}{2}}\, c(t)$ "$-$" from reversed integration $t \ldots t_1$

SUBTRACT Eq II above gives:

Input & output relation.

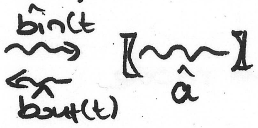
$$\boxed{\hat{b}_{out}(t) - \hat{b}_{in}(t) = \sqrt{\gamma}\, \hat{a}}$$

more general

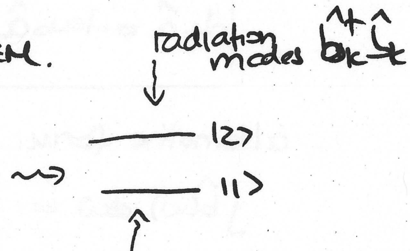
$$\hat{b}_{out} - \hat{b}_{in} = \sqrt{\gamma}\, \hat{c}$$

Input-Output theory of open quantum systems

Example: cavity or resonator



Quantum Langevin Equations for a TWO LEVEL SYSTEM.



A two level system has states $\{|1\rangle, |2\rangle\}$ Basis

define Pauli-Operators / Matrices

$\hat\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ thus $\hat\sigma_z|1\rangle = |1\rangle$ $\quad \hat\sigma_z|2\rangle = -1|2\rangle$ $\quad \hat\sigma_z|1\rangle = \hat\sigma_z|2\rangle = 0$

$\hat\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ raising operator (i.e $\sigma^+ \equiv |2\rangle\langle 1|$) $\quad \hat\sigma^+|1\rangle = |2\rangle$ $\quad \hat\sigma^+|2\rangle = 0$;

$\hat\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ lowering operator ($\sigma^- \equiv |1\rangle\langle 2|$) $\quad \hat\sigma^-|2\rangle = |1\rangle$ $\quad \hat\sigma^-|1\rangle = 0$!

Pauli Matrices $\hat\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\hat\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Thus $\hat\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y)$ $\qquad [\hat\sigma_-, \hat\sigma_+] = -\hat\sigma_z$

$\hat\sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y)$ $\qquad [\hat\sigma_-, \hat\sigma_z] = 2\hat\sigma_-$

commutator relations

let use assume the Two level system is coupled to an electromagnetic field (infinitely many modes). The Model:

$$\boxed{\begin{aligned} \hat H_{sys} &= \frac{1}{2}\hbar\Omega\hat\sigma_z \\ H_{BATH} &= \sum \hbar\omega_k \hat b_k^+ \hat b_k \\ \hat H_{int} &= \sum \hbar g_k (\hat\sigma_- b_k^+ + \hat\sigma_+ \hat b_k) \end{aligned}}$$

absorption and emission events.

This theory can successfully describe SPONTANEOUS EMISSION an irreversible process qualitatively correct.

Apply same procedure [homework]

$$\boxed{\begin{aligned} d_t\hat\sigma_- &= -\frac{i}{\hbar}[\hat\sigma_-, \hat H] = -i\Omega\hat\sigma_- - \frac{\kappa}{2}\hat\sigma^- + \sqrt{\kappa}\,\underbrace{\hat\sigma_z \hat b_{in}(t)}_{\text{NON LINEAR TERM}} \\ d_t\hat\sigma_+ &= (+i\Omega - \kappa/2)\,\hat\sigma_+ + \sqrt{\kappa}\,\underbrace{\hat\sigma_z \hat b_{in}^+(t)}_{\text{NONLINEAR TERM}} \\ d_t\hat\sigma_z &= -\kappa(1+\hat\sigma_z) - 2\sqrt{\kappa}\,\{\hat\sigma^+ \hat b_{in}(t) + \hat b_{in}^+(t)\hat\sigma^-\} \end{aligned}}$$

decays with κ

Thus the equations are NOT LINEAR since $\hat\sigma_-$ depends on $\langle\hat\sigma_z \hat b_{in}\rangle$

the equations are not closed. However the equation is simpler if the atom is initially in excited state $|2\rangle$ and fields are in vacuum state $\langle \hat{b}^+(0)\hat{b}(0)\rangle = 0$ at time $t = t_0$ and $\langle \hat{b}_{in}\rangle = 0$

then:

$$\boxed{\begin{aligned} \langle \dot{\hat{\sigma}}_-\rangle &= (-i\Omega - \kappa/2)\langle \hat{\sigma}_-\rangle \\ \langle \dot{\hat{\sigma}}_+\rangle &= (+i\Omega - \kappa/2)\langle \hat{\sigma}_+\rangle \\ \langle \dot{\hat{\sigma}}_z\rangle &= -\kappa(1 + \langle \hat{\sigma}_z\rangle) \end{aligned}}$$

BLOCH EQS (WITHOUT DRIVING FIELD)

where we need to assume that $\langle \hat{\sigma}_z \hat{b}_{in}(t)\rangle = \langle \hat{\sigma}_z\rangle \underbrace{\langle \hat{b}_{in}(t)\rangle}_{=0} = 0$

(i.e the system is in a separable state)

Hence the atom decays:

$$\langle \hat{\sigma}_z(t)\rangle = \hat{\sigma}_z(0) \cdot e^{-\kappa t}$$

and $\kappa = |g_k|^2 \cdot 2\pi\, D(\omega)$ $\quad$ $D(\omega)$ density of states of the electromagnetic field

Thus we have a rigorous expression for the atomic decay $\quad D(\omega) = \dfrac{\omega^2}{c^3\pi^2}$

$$\kappa = |g_k|^2 \frac{\omega^2}{c^3\pi^2} 2\pi$$

$|g_k|^2$: dipole coupling coefficient.

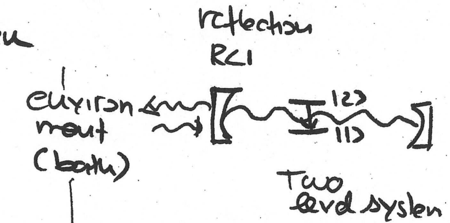
Note: This so called "Wigner Weisskopf theory of spont. emission"

a quantum vacuum effect of the [illegible] field

Reference: Scully Quantum Optics Chapter 9

OTHER Example: Nobel Prize 2015 S. Haroche (cavity QED, quantum electrodynamics)

Two level system inside a cavity that is open



cavity mode:	$\hat{H} = a^+a\hbar\omega$
output modes	$\hat{H}_{Bath} = \sum_k \omega_k \hbar\, b_k^+ b_k$
Interaction	$\hat{H}_{int}^{atom} = \hbar \sum g_k^a (\hat{\sigma}^+ \hat{a} + \hat{\sigma}^- \hat{a}^+)$
Interaction	$\hat{H}_{int}^{cav} = \hbar \sum g_k^c (\hat{a}\hat{b}_k^+ + \hat{a}^+\hat{b}_k)$

Note. Here we neglect coupling of the atom to outside EM-modes.

Driving fields: (Input fields)

In particular in the case of two level systems there is often a classical external driving field. Thus one of the bath modes is serving as an input or output mode. To achieve this we replace $\langle \hat{b}_{in}(t)\rangle$ by a non-zero quantity

$$\langle \hat{b}_{in}(t)\rangle = b(t) \neq 0$$

This is equivalent to the system Hamiltonian: $\hat{H}_{drive} = 2\, b(t)\, \hbar\, (\hat{a}^+ + \hat{a})$

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(ref. Scully chapter 9)

Using the elimination method of the bath modes we arrive at

$$\left|\begin{array}{l}\frac{d\langle\hat a^\dagger\hat a\rangle}{dt} = ig\langle\hat\sigma_+\hat a - \hat a^\dagger\hat\sigma_-\rangle - \kappa\langle\hat a^\dagger\hat a\rangle + \kappa\bar n_{th}\\ \frac{d}{dt}\langle\hat\sigma_z\rangle = -2ig\langle\hat\sigma_+\hat a - \hat a^\dagger\hat\sigma_-\rangle\end{array}\right|$$

The brackets $\langle..\rangle$ denote both expectations of the operators of bath modes and operator system modes.

again the system is NOT CLOSED and thus cannot be solved without approximations.

Simple situation : optical modes / fields ($\hbar\omega \gg k_BT \Rightarrow \langle a^\dagger a\rangle = 0$)
i.e $\bar n_{TH} \simeq 0$

The equation of motion of $\langle\hat\sigma_+\hat a - \hat\sigma_-\hat a^\dagger\rangle$ will depend on $\langle\hat a^\dagger\hat\sigma_z\hat a\rangle$ and thus lead to one more order, indeed

$$\frac{d}{dt}[\hat a^{\dagger m}\hat a^n\hat O_{ATOMIC}] = \frac{-i}{\hbar}[\hat a^{\dagger m}\hat a^n\hat O_{AT.}, \hat H_{tot}] + \frac{d}{dt}\langle\frac{d}{dt}[a^{\dagger m}a^n]\rangle\hat O_{ATOMIC}$$

$\hat O_{atomic} = \{\sigma_+, \sigma_-, \hat\sigma_z\}$

If the environment is in the vacuum state $\hat b_k|0\rangle = 0$ ($\bar n_{TH} = 0$) and cavity is in vacuum $|0\rangle$:

$$\langle\hat a^{\dagger 2}\hat\sigma_z\hat a^2\rangle = \langle 0_a 1|\hat a^\dagger\hat\sigma_z\hat a^2|1\,0_a\rangle = \langle 1|\hat\sigma_z|1\rangle\underbrace{\langle 0_a|\hat a^{\dagger 2}\hat a^2|0_a\rangle}_{=0}$$

thus all higher orders vanish.

One arrives at

$$\begin{aligned} d_t\langle\hat a^\dagger\hat a\rangle &= g\hat A_1 - \kappa\langle\hat a^\dagger\hat a\rangle \\ d_t\langle\hat\sigma_z\rangle &= -2g\hat A_1 \\ d_t\hat A_1 &= g\langle\hat\sigma_z\rangle + 2g\hat A_2 + g - \kappa/2\,\hat A_1 \\ d_t\hat A_2 &= -g\hat A_1 - \kappa\hat A_2 \end{aligned}$$

$\hat A_1 \equiv i\langle\hat\sigma_+\hat a - \hat a^\dagger\hat\sigma_-\rangle$

$\hat A_2 \equiv \langle\hat a^\dagger\hat\sigma_z\hat a\rangle$

Solution of the atom is

$$\boxed{\langle\sigma_z(t)\rangle = -1 + 2e^{-4g^2t/\kappa}}$$

PURCELL EFFECT (SPONTANEOUS EMISSION ENHANCEMENT)

I.e the atomic decay is

$$\Gamma_c = \frac{4g^2}{\kappa}$$

PURCELL EFFECT.

in physical quantities:

$$\left(\text{i.e } \Gamma_c = \frac{1}{4\pi\epsilon_0}\frac{4\omega^3\wp_{ab}}{3\hbar c^3}\left(\frac{\nu}{\kappa}\right)\left(\frac{6\pi c^3}{V\nu^3}\right)\right)$$

Quantum Regression Theorem for a complete set of operators

Key insight: The equations of motion for the expectation values of system operators (ONE-TIME AVERAGE) are also equations of motions for the TWO-TIME AVERAGE, ie correlations.

Why do we need TWO-TIME AVERAGES? to evaluate output spectra e.g.

$$\boxed{S_{aa}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle a^+(t)\, a(t+\tau)\rangle\, e^{i\omega\tau}\, d\tau} \quad \text{SPECTRUM}$$

or e.g in quantum optics correlation of photons. For example intensity autocorrelation function $g^2(\tau)$

$$g^2(\tau) \propto \langle \hat{a}^+(t)\, \hat{a}^+(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle$$

For a complete set of operators $\hat{A}_\mu$ $\mu = 1,2\ldots$ (system operators)

$$\boxed{\langle \dot{\hat{A}}_\mu\rangle = \sum_\lambda \underbrace{M_{\lambda\mu}}_{\text{Matrix (square)}} \langle A_\mu\rangle} \quad (*) \quad \text{(ONE-TIME AVERAGES)}$$

ie closed QLE.

i.e $\langle \dot{\vec{A}}\rangle = \underline{\underline{M}}\, \langle \vec{A}\rangle$ in vector notation.

> For the TWO-TIME AVERAGES. (for operators $\hat{O}$ and $\hat{A}$ of the system) we have for a complete set of operators obeying (*):
>
> $$\frac{d}{d\tau}\langle \hat{O}_i(t)\,\hat{A}_\ell(t+\tau)\rangle = \sum_j M_{ij}\langle \hat{O}_j\, \hat{A}_\ell(t+\tau)\rangle$$

QUANTUM REGRESSION THEOREM

Applications: Fluorescence spectrum of atoms, sensitivity limit ie standard quantum limit of LIGO; photon counting statistics ....

Applying the Quantum regression theorem, example system operators $\hat{A}_\mu = \{\hat{a}, \hat{a}^+\}$

$\hat{A}_1 = \hat{a}$ and $\hat{A}_2 = \hat{a}^+$ thus: $M_{11} = (-\gamma/2 + i\omega)$ since:

$$\partial_t\langle\hat{a}\rangle = -i\omega\langle\hat{a}\rangle - \gamma/2\,\langle\hat{a}\rangle + \underbrace{\langle\hat{f}\rangle}_{=0}$$

$$\underline{\underline{M}} = \begin{pmatrix} i\omega - \gamma/2 & 0 \\ 0 & -i\omega - \gamma/2 \end{pmatrix} \qquad M_{22} = (-\gamma/2 - i\omega)$$

$$\frac{d}{d\tau}\langle \hat{a}^+(t)\,\hat{a}(t+\tau)\rangle = \frac{d}{d\tau}\langle \hat{A}_1\hat{O}_1\rangle = \sum_j M_{1j}\langle \hat{A}_j O_1\rangle = M_{11}\langle \hat{A}_1\hat{O}_1\rangle$$

$$\Leftrightarrow \quad \frac{d}{d\tau}\langle \hat{a}^+(t)\,\hat{a}(t+\tau)\rangle = (i\omega - \gamma/2)\langle\hat{a}^+(t)\,\hat{a}(t+\tau)\rangle$$

$$\Leftrightarrow \quad \boxed{\langle \hat{a}^+(t)\,\hat{a}(t+\tau)\rangle = \langle\hat{a}^+(t)\,\hat{a}(t)\rangle\, e^{(i\omega_0 - \gamma/2)\tau}}$$

Lorentzian spectrum as before $\langle\hat{a}^+(t)\,\hat{a}(t)\rangle \to \bar{n}$